10-4 Ellipses

Main Ideas

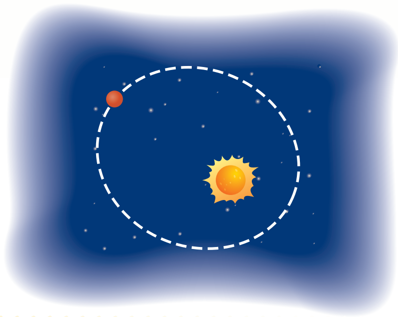
- Write equations of ellipses.
- Graph ellipses.

New Vocabulary

ellipse
foci
major axis
minor axis
center

GET READY for the Lesson

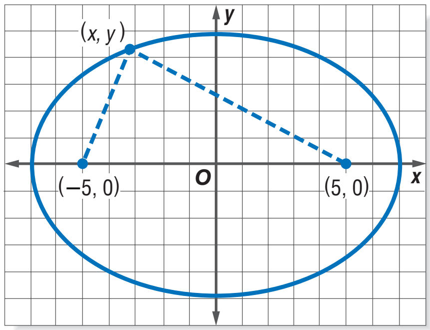
Fascination with the sky has caused people to wonder, observe, and make conjectures about the planets since the beginning of history. Since the early 1600s, the orbits of the planets have been known to be ellipses with the Sun at a focus.



Equations of Ellipses

As you discovered in the Algebra Lab on page 580, an **ellipse** is the set of all points in a plane such that the sum of the distances from two fixed points is constant. The two fixed points are called the **foci** of the ellipse.

The ellipse at the right has foci at (5, 0) and (−5, 0). The distances from either of the x-intercepts to the foci are 2 units and 12 units, so the sum of the distances from any point with coordinates (x, y) on the ellipse to the foci is 14 units.



You can use the Distance Formula and the definition of an ellipse to find an equation of this ellipse.

Study Tip

Ellipses
In an ellipse, the constant sum that is the distance from two fixed points must be greater than the distance between the foci.

$$\underbrace{\sqrt{(x+5)^2+y^2}}_{\text{The distance between } (x,\,y) \text{ and } (-5,\,0)} + \underbrace{\sqrt{(x-5)^2+y^2}}_{\text{the distance between } (x,\,y) \text{ and } (5,\,0)} = 14.$$

$$\sqrt{(x+5)^2+y^2} + \sqrt{(x-5)^2+y^2} = 14$$

$$\sqrt{(x+5)^2+y^2} = 14 - \sqrt{(x-5)^2+y^2} \qquad \text{Isolate the radicals.}$$

$$(x+5)^2+y^2 = 196 - 28\sqrt{(x-5)^2+y^2} + (x-5)^2+y^2$$

$$x^2+10x+25+y^2 = 196 - 28\sqrt{(x-5)^2+y^2} + x^2-10x+25+y^2$$

$$20x - 196 = -28\sqrt{(x-5)^2+y^2} \qquad \text{Simplify.}$$

$$5x - 49 = -7\sqrt{(x-5)^2+y^2} \qquad \text{Divide each side by 4.}$$

$$25x^2 - 490x + 2401 = 49[(x-5)^2+y^2] \qquad \text{Square each side.}$$

$$25x^2 - 490x + 2401 = 49x^2 - 490x + 1225 + 49y^2 \qquad \text{Distributive Property}$$

$$-24x^2 - 49y^2 = -1176 \qquad \text{Simplify.}$$

$$\frac{x^2}{49} + \frac{y^2}{24} = 1 \qquad \text{Divide each side by } -1176.$$

An equation for this ellipse is $\frac{x^2}{49} + \frac{y^2}{24} = 1$.

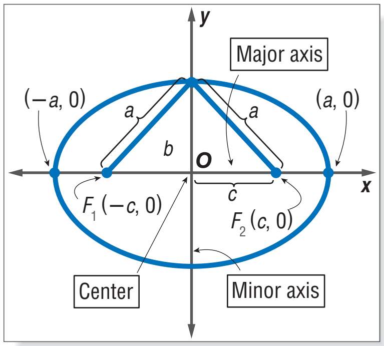
Study Tip

Vertices of Ellipses

The endpoints of each axis are called the *vertices* of the ellipse.

Every ellipse has two axes of symmetry. The points at which the ellipse intersects its axes of symmetry determine two segments with endpoints on the ellipse called the **major axis** and the **minor axis.** The axes intersect at the **center** of the ellipse. The foci of an ellipse always lie on the major axis.

Study the ellipse at the right. The sum of the distances from the foci to any point on the ellipse is the same as the length of the major axis, or $2a$ units. The distance from the center to either focus is c units. By the Pythagorean Theorem, a, b, and c are related by the equation $c^2 = a^2 - b^2$. Notice that the x- and y-intercepts, $(\pm a, 0)$ and $(0, \pm b)$, satisfy the quadratic equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. This is the standard form of the equation of an ellipse with its center at the origin and a horizontal major axis.



Study Tip

Equations of Ellipses

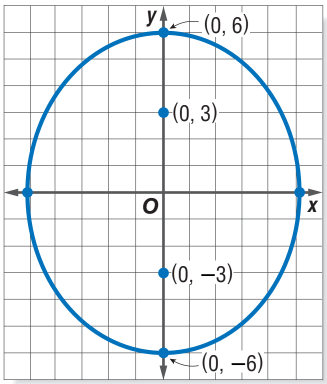
In either case, $a^2 \geq b^2$ and $c^2 = a^2 - b^2$. You can determine if the foci are on the x-axis or the y-axis by looking at the equation. If the x^2 term has the greater denominator, the foci are on the x-axis. If the y^2 term has the greater denominator, the foci are on the y-axis.

KEY CONCEPT *Equations of Ellipses with Centers at the Origin*

Standard Form of Equation	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$
Direction of Major Axis	horizontal	vertical
Foci	$(c, 0), (-c, 0)$	$(0, c), (0, -c)$
Length of Major Axis	$2a$ units	$2a$ units
Length of Minor Axis	$2b$ units	$2b$ units

EXAMPLE Write an Equation for a Graph

1 **Write an equation for the ellipse.**



To write the equation for the ellipse, we need to find the values of a and b for the ellipse. We know that the length of the major axis of any ellipse is $2a$ units. In this ellipse, the length of the major axis is the distance between the points at (0, 6) and (0, −6). This distance is 12 units.

$2a = 12$ Length of major axis = 12

$a = 6$ Divide each side by 2.

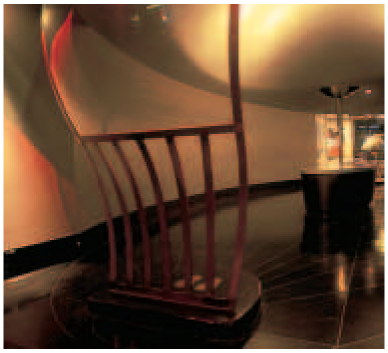
The foci are located at (0, 3) and (0, −3), so $c = 3$. We can use the relationship between a, b, and c to determine the value of b.

$c^2 = a^2 - b^2$ Equation relating a, b, and c

$9 = 36 - b^2$ $c = 3$ and $a = 6$

$b^2 = 27$ Solve for b^2.

Since the major axis is vertical, substitute 36 for a^2 and 27 for b^2 in the form $\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$. An equation of the ellipse is $\frac{y^2}{36} + \frac{x^2}{27} = 1$.



Real-World Link

The whispering gallery at Chicago's Museum of Science and Industry has a parabolic dish at each focus to help collect sound.

Source: msichicago.org

CHECK Your Progress

1. Write an equation for the ellipse with endpoints of the major axis at $(-5, 0)$ and $(5, 0)$ and endpoints of the minor axis at $(0, -2)$ and $(0, 2)$.

EXAMPLE 2 Write an Equation Given the Lengths of the Axes

MUSEUMS In an ellipse, sound or light coming from one focus is reflected to the other focus. In a whispering gallery, a person can hear another person whisper from across the room if the two people are standing at the foci. The whispering gallery at the Museum of Science and Industry in Chicago has an elliptical cross section that is 13 feet 6 inches by 47 feet 4 inches.

a. Write an equation to model this ellipse. Assume that the center is at the origin and the major axis is horizontal.

The length of the major axis is $47\frac{1}{3}$ or $\frac{142}{3}$ feet.

$2a = \frac{142}{3}$ Length of major axis $= \frac{142}{3}$

$a = \frac{71}{3}$ Divide each side by 2.

The length of the minor axis is $13\frac{1}{2}$ or $\frac{27}{2}$ feet.

$2b = \frac{27}{2}$ Length of minor axis $= \frac{27}{2}$

$b = \frac{27}{4}$ Divide each side by 2.

Substitute $a = \frac{71}{3}$ and $b = \frac{27}{4}$ into the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. An equation of the ellipse is $\frac{x^2}{\left(\frac{71}{3}\right)^2} + \frac{y^2}{\left(\frac{27}{4}\right)^2} = 1$.

b. How far apart are the points at which two people should stand to hear each other whisper?

People should stand at the two foci of the ellipse. The distance between the foci is $2c$ units.

$c^2 = a^2 - b^2$ Equation relating a, b, and c

$c = \sqrt{a^2 - b^2}$ Take the square root of each side.

$2c = 2\sqrt{a^2 - b^2}$ Multiply each side by 2.

$2c = 2\sqrt{\left(\frac{71}{3}\right)^2 - \left(\frac{27}{4}\right)^2}$ Substitute $a = \frac{71}{3}$ and $b = \frac{27}{4}$.

$2c \approx 45.37$ Use a calculator.

The points where two people should stand to hear each other whisper are about 45.37 feet or about 45 feet 4 inches apart.

CHECK Your Progress

BILLIARDS Elliptipool is an elliptical pool table with only one pocket that is located on one of the foci. If the ball is placed on the other focus and shot off any edge, it will drop into the pocket located on the other focus. The pool table has axes that are 4 feet 6 inches and 5 feet.

2A. Write an equation to model this ellipse. Assume that the center is at the origin and the major axis is horizontal.

2B. How far apart are the two foci?

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Ray F. Hillstrom, Jr.

Graph Ellipses As with circles, you can use completing the square, symmetry, and transformations to help graph ellipses. An ellipse with its center at the origin is represented by an equation of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ or $\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$.

> **Study Tip**
>
> **Graphing Calculator**
>
> You can graph an ellipse on a graphing calculator by first solving for y. Then graph the two equations that result on the same screen.

EXAMPLE Graph an Equation in Standard Form

3 **Find the coordinates of the center and foci and the lengths of the major and minor axes of the ellipse with equation $\frac{x^2}{16} + \frac{y^2}{4} = 1$. Then graph the ellipse.**

The center of this ellipse is at (0, 0). Since $a^2 = 16$, $a = 4$. Since $b^2 = 4$, $b = 2$.

The length of the major axis is 2(4) or 8 units, and the length of the minor axis is 2(2) or 4 units. Since the x^2 term has the greater denominator, the major axis is horizontal.

$c^2 = a^2 - b^2$ Equation relating a, b, and c

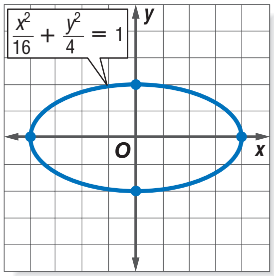
$c^2 = 4^2 - 2^2$ or 12 $a = 4, b = 2$

$c = \sqrt{12}$ or $2\sqrt{3}$ Take the square root of each side.

The foci are at $(2\sqrt{3}, 0)$ and $(-2\sqrt{3}, 0)$.

You can use a calculator to find some approximate nonnegative values for x and y that satisfy the equation. Since the ellipse is centered at the origin, it is symmetric about the y-axis. Therefore, the points at (–4, 0), (–3, 1.3), (–2, 1.7), and (–1, 1.9) lie on the graph.

The ellipse is also symmetric about the x-axis, so the points at (–3, –1.3), (–2, –1.7), (–1, –1.9), (0, –2), (1, –1.9), (2, –1.7), and (3, –1.3) lie on the graph.



Graph the intercepts, (–4, 0), (4, 0), (0, 2), and (0, –2), and draw the ellipse that passes through them and the other points.

CHECK Your Progress

3. Find the coordinates of the foci and the lengths of the major and minor axes of the ellipse with equation $\frac{x^2}{49} + \frac{y^2}{36} = 1$. Then graph the ellipse.

Suppose an ellipse is translated h units right and k units up, moving the center to the point (h, k). Such a move would be equivalent to replacing x with $x - h$ and replacing y with $y - k$.

KEY CONCEPT *Equations of Ellipses with Centers at (h, k)*

Standard Form of Equation	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$
Direction of Major Axis	horizontal	vertical
Foci	$(h \pm c, k)$	$(h, k \pm c)$

EXAMPLE Graph an Equation Not in Standard Form

4 **Find the coordinates of the center and foci and the lengths of the major and minor axes of the ellipse with equation $x^2 + 4y^2 + 4x - 24y + 24 = 0$. Then graph the ellipse.**

Complete the square for each variable to write this equation in standard form.

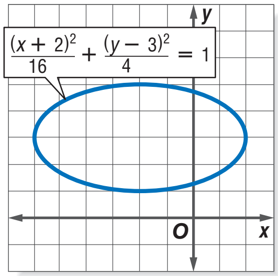
$x^2 + 4y^2 + 4x - 24y + 24 = 0$	Original equation
$(x^2 + 4x + \blacksquare) + 4(y^2 - 6y + \blacksquare) = -24 + \blacksquare + 4(\blacksquare)$	Complete the squares.
$(x^2 + 4x + 4) + 4(y^2 - 6y + 9) = -24 + 4 + 4(9)$	$\left(\frac{4}{2}\right)^2 = 4$, $\left(\frac{-6}{2}\right)^2 = 9$
$(x + 2)^2 + 4(y - 3)^2 = 16$	Write the trinomials as perfect squares.
$\frac{(x + 2)^2}{16} + \frac{(y - 3)^2}{4} = 1$	Divide each side by 16.

Study Tip

Look Back

To review **grouping** and **factoring** common factors of each variable separately, see Lessons 5-3 and 5-5.

The graph of this ellipse is the graph from Example 3 translated 2 units to the left and up 3 units. The center is at $(-2, 3)$ and the foci are at $(-2 + 2\sqrt{3}, 0)$ and $(-2 - 2\sqrt{3}, 0)$. The length of the major axis is still 8 units, and the length of the minor axis is still 4 units.



CHECK Your Progress

4. Find the coordinates and foci and the lengths of the major and minor axes of the ellipse with equation $x^2 + 6y^2 + 8x - 12y + 16 = 0$. Then graph the ellipse.

You can use a circle to locate the foci on the graph of a given ellipse.

ALGEBRA LAB

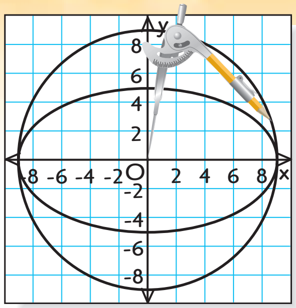
Locating Foci

Step 1 Graph an ellipse so that its center is at the origin. Let the endpoints of the major axis be at $(-9, 0)$ and $(9, 0)$, and let the endpoints of the minor axis be at $(0, -5)$ and $(0, 5)$.

Step 2 Use a compass to draw a circle with center at $(0, 0)$ and radius 9 units.

Step 3 Draw the line with equation $y = 5$ and mark the points at which the line intersects the circle.

Step 4 Draw perpendicular lines from the points of intersection to the x-axis. The foci of the ellipse are located at the points where the perpendicular lines intersect the x-axis.



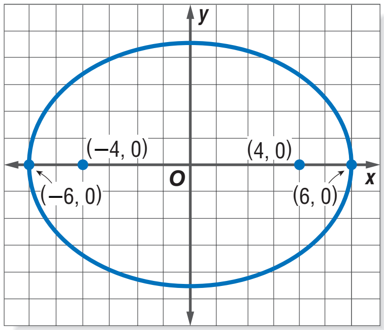
MAKE A CONJECTURE

Draw another ellipse and locate its foci. Why does this method work?

CHECK Your Understanding

Example 1 (pp. 582–583)

1. Write an equation for the ellipse shown at the right.



Write an equation for the ellipse that satisfies each set of conditions.

2. endpoints of major axis at (2, 2) and (2, −10), endpoints of minor axis at (0, −4) and (4, −4)

3. endpoints of major axis at (0, 10) and (0, −10), foci at (0, 8) and (0, −8)

Example 2 (p. 583)

4. ASTRONOMY At its closest point, Earth is 0.99 astronomical units from the center of the Sun. At its farthest point, Earth is 1.021 astronomical units from the center of the Sun. Write an equation for the orbit of Earth, assuming that the center of the orbit is the origin and the Sun lies on the x-axis.

Examples 3, 4 (pp. 584–585)

Find the coordinates of the center and foci and the lengths of the major and minor axes for the ellipse with the given equation. Then graph the ellipse.

5. $\frac{y^2}{18} + \frac{x^2}{9} = 1$

6. $\frac{(x-1)^2}{20} + \frac{(y+2)^2}{4} = 1$

7. $4x^2 + 8y^2 = 32$

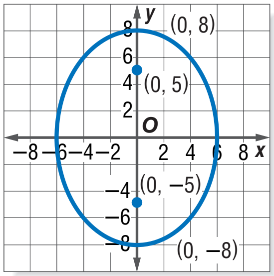
8. $x^2 + 25y^2 - 8x + 100y + 91 = 0$

Exercises

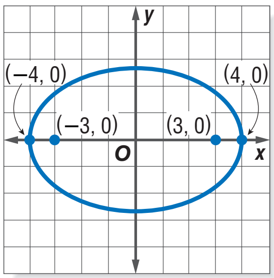
HOMEWORK HELP	
For Exercises	See Examples
9–15	1
16, 17	2
18–21	3
22–25	4

Write an equation for each ellipse.

9.



10.



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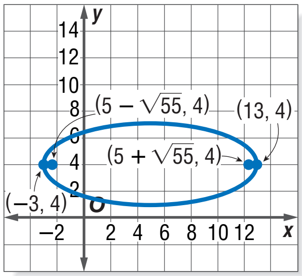
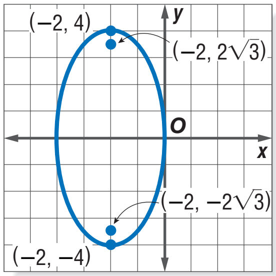


12.



Write an equation for the ellipse that satisfies each set of conditions.

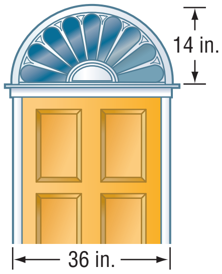
13. endpoints of major axis at (−11, 5) and (7, 5), endpoints of minor axis at (−2, 9) and (−2, 1)

14. endpoints of major axis at (2, 12) and (2, −4), endpoints of minor axis at (4, 4) and (0, 4)

15. major axis 20 units long and parallel to y-axis, minor axis 6 units long, center at (4, 2)

16. **ASTRONOMY** At its closest point, Venus is 0.719 astronomical units from the Sun. At its farthest point, Venus is 0.728 astronomical units from the Sun. Write an equation for the orbit of Venus. Assume that the center of the orbit is the origin, the Sun lies on the x-axis, and the radius of the Sun is 400,000 miles.

17. **INTERIOR DESIGN** The rounded top of the window is the top half of an ellipse. Write an equation for the ellipse if the origin is at the midpoint of the bottom edge of the window.



Find the coordinates of the center and foci and the lengths of the major and minor axes for the ellipse with the given equation. Then graph the ellipse.

18. $\frac{y^2}{10} + \frac{x^2}{5} = 1$

19. $\frac{x^2}{25} + \frac{y^2}{9} = 1$

20. $\frac{(x+8)^2}{144} + \frac{(y-2)^2}{81} = 1$

21. $\frac{(y+11)^2}{144} + \frac{(x-5)^2}{121} = 1$

22. $3x^2 + 9y^2 = 27$

23. $27x^2 + 9y^2 = 81$

24. $7x^2 + 3y^2 - 28x - 12y = -19$

25. $16x^2 + 25y^2 + 32x - 150y = 159$

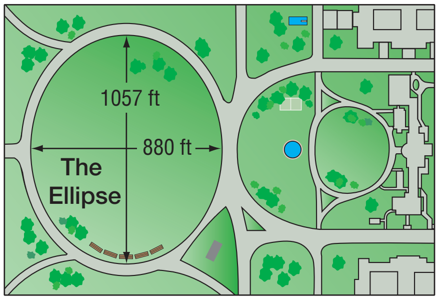
Real-World Link

The Ellipse, also known as President's Park South, has an area of about 16 acres.

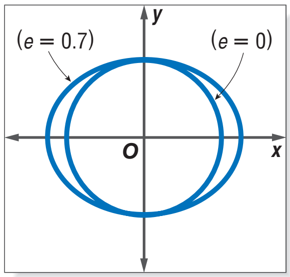
Write an equation for the ellipse that satisfies each set of conditions.

26. major axis 16 units long and parallel to x-axis, minor axis 9 units long, center at (5, 4)
27. endpoints of major axis at (10, 2) and (−8, 2), foci at (6, 2) and (−4, 2)
28. endpoints of minor axis at (0, 5) and (0, −5), foci at (12, 0) and (−12, 0)
29. Write the equation $10x^2 + 2y^2 = 40$ in standard form.
30. What is the standard form of the equation $x^2 + 6y^2 - 2x + 12y - 23 = 0$?

31. **WHITE HOUSE** There is an open area south of the White House known as The Ellipse. Write an equation to model The Ellipse. Assume that the origin is at the center of The Ellipse.



32. **ASTRONOMY** In an ellipse, the ratio $\frac{c}{a}$ is called the eccentricity and is denoted by the letter e. Eccentricity measures the elongation of an ellipse. The closer e is to 0, the more an ellipse looks like a circle. Pluto has the most eccentric orbit in our solar system with $e \approx 0.25$. Find an equation to model the orbit of Pluto, given that the length of the major axis is about 7.34 billion miles. Assume that the major axis is horizontal and that the center of the orbit is the origin.



EXTRA PRACTICE
See pages 912, 935.
Math Online
Self-Check Quiz at algebra2.com

H.O.T. Problems

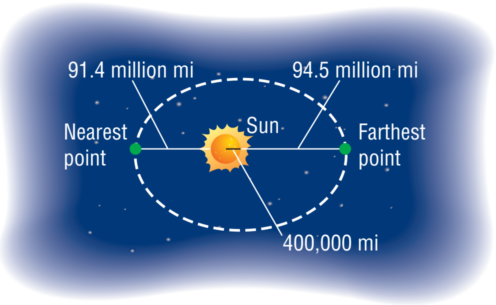
33. **REASONING** Explain why a circle is a special case of an ellipse.

34. **OPEN ENDED** Write an equation for an ellipse with its center at (2, −5) and a horizontal major axis.

James P. Blair/CORBIS

35. **CHALLENGE** Find an equation for the ellipse with foci at $(\sqrt{3}, 0)$ and $(-\sqrt{3}, 0)$ that passes through $(0, 3)$.

36. ***Writing in Math*** Use the information about the solar system on page 581 and the figure at the right to explain why ellipses are important in the study of the solar system. Explain why an equation that is an accurate model of the path of a planet might be useful.



STANDARDIZED TEST PRACTICE

37. **ACT/SAT** Winona is making an elliptical target for throwing darts. She wants the target to be 27 inches wide and 15 inches high. Which equation should Winona use to draw the target?

A $\frac{y^2}{13.5} + \frac{x^2}{7.5} = 1$

B $\frac{y^2}{182.25} + \frac{x^2}{56.25} = 1$

C $\frac{y^2}{56.25} + \frac{x^2}{182.25} = 1$

D $\frac{y^2}{7.5} + \frac{x^2}{13.5} = 1$

38. **REVIEW** What is the standard form of the equation of the conic given below?

$$2x^2 - 4y^2 - 8x - 24y - 16 = 0$$

F $\frac{(x-4)^2}{11} - \frac{(y+3)^2}{3} = 1$

G $\frac{(y-3)^2}{3} - \frac{(x-2)^2}{6} = 1$

H $\frac{(y+3)^2}{4} - \frac{(x+2)^2}{5} = 1$

J $\frac{(x-4)^2}{11} + \frac{(y+3)^2}{3} = 1$

Spiral Review

Write an equation for the circle that satisfies each set of conditions. (Lesson 10-3)

39. center $(3, -2)$, radius 5 units

40. endpoints of a diameter at $(5, -9)$ and $(3, 11)$

41. Write an equation of a parabola with vertex $(3, 1)$ and focus $\left(3, 1\frac{1}{2}\right)$. Then draw the graph. (Lesson 10-2)

MARRIAGE For Exercises 42–44, use the table below that shows the number of married Americans over the last few decades. (Lesson 2-5)

Year	1980	1990	1995	1999	2000	2010
People (millions)	104.6	112.6	116.7	118.9	120.2	?

Source: U.S. Census Bureau

42. Draw a scatter plot in which x is the number of years since 1980.

43. Find a prediction equation.

44. Predict the number of married Americans in 2010.

GET READY for the Next Lesson

PREREQUISITE SKILL Graph the line with the given equation. (Lessons 2-1, 2-2, and 2-3)

45. $y = 2x$

46. $y = -2x$

47. $y = -\frac{1}{2}x$

48. $y = \frac{1}{2}x$

49. $y + 2 = 2(x - 1)$

50. $y + 2 = -2(x - 1)$